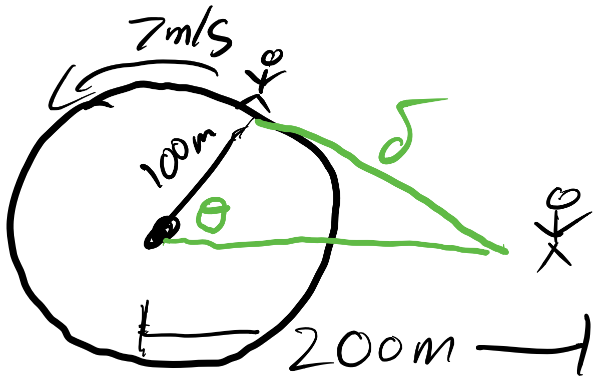
Section 3.10 – Related Rates; Conclusion

Ex: A runner sprints along a circular track with radius 100 m at constant speed of 7 m/s. A friend stands 200 m from the center of the track. How fast is the distance between the friends changing when they are 200 meters apart?



Want: $\left.\frac{d\delta}{dt}\right|_{\delta=200m}$

Know: $\frac{d\theta}{dt} = 0.07 \text{ rad/s}$

(Because $7\cancel{m}/s \cdot 1\text{ rad}/100\cancel{m} = \frac{7}{100}$ rad/s)

Need: Relationship btwn δ and θ.

Law of Cosines says $\delta^2 = 100^2 + 200^2 - 2\cdot 100 \cdot 200 \cos\theta$

Diff on both sides: $2\delta \frac{d\delta}{dt} = -40{,}000(-\sin\theta)\frac{d\theta}{dt}$

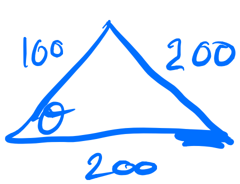
$= 40000 \sin\theta\,(0.07)$

$= 2800 \sin\theta$

Plug in $\delta = 200$ m $\Rightarrow 400\left(\left.\frac{d\delta}{dt}\right|_{\delta=200}\right) = 2800 \sin\theta\big|_{\delta=200}$

$\left.\frac{d\delta}{dt}\right|_{\delta=200} = 7 \sin\theta\big|_{\delta=200}$

$\cancel{200^2} = 100^2 + \cancel{200^2} - 2\cdot\cancel{100}\cdot 200 \cdot \cos\theta$



$400\cos\theta = 100 \Leftrightarrow \cos\theta = \frac{1}{4}$

$\theta = \pm 1.32$ rad

$\sin\theta = \pm 0.97$

$\frac{d\delta}{dt}\Big|_{\delta=200} = 7(\pm 0.97) = \boxed{\pm 6.78 \text{ m/s}}$

Section 4.2 – Extreme Values

Big Idea: Using calculus, we can find the points $(x, f(x))$ which are highest up (largest y coordinate) on the graph of f. Similarly, we can find the lowest points on the graph.

- Very useful! For example, suppose $p(x)$ represents Apple's profit yield for producing x iPads. These techniques show Apple the number of iPads to produce in order to maximize profit!

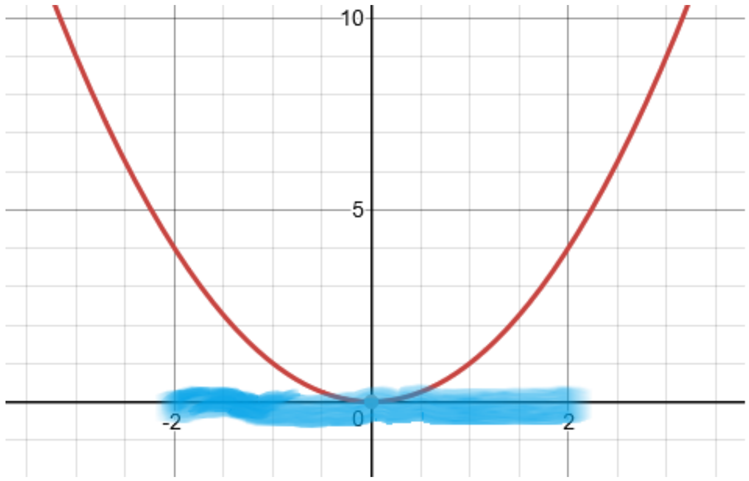
Definition: Let f be a function defined on an interval I. If a is an element of I, then we say that:

(i) $f(a)$ is the absolute minimum of f on I if $f(a) \leq f(x)$ for all x in I.

(ii) $f(a)$ is the absolute maximum of f on I if $f(a) \geq f(x)$ for all x in I.

Vocabulary Remark: One often says instead of (i) that "f on I has an absolute minimum at a." Similarly for (ii).

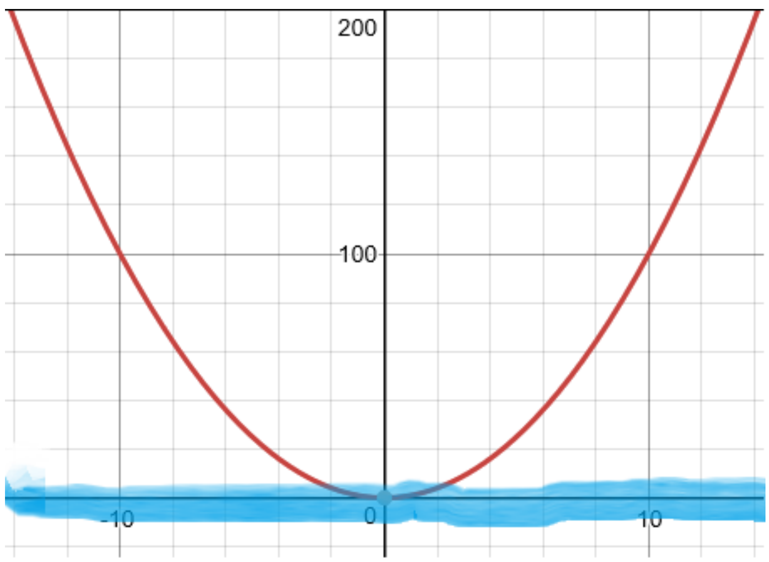
Absolute maxima and minima need not exist. Here are examples of several of the possibilities that can occur.



$f(x) = x^2$

$I = [-2, 2]$

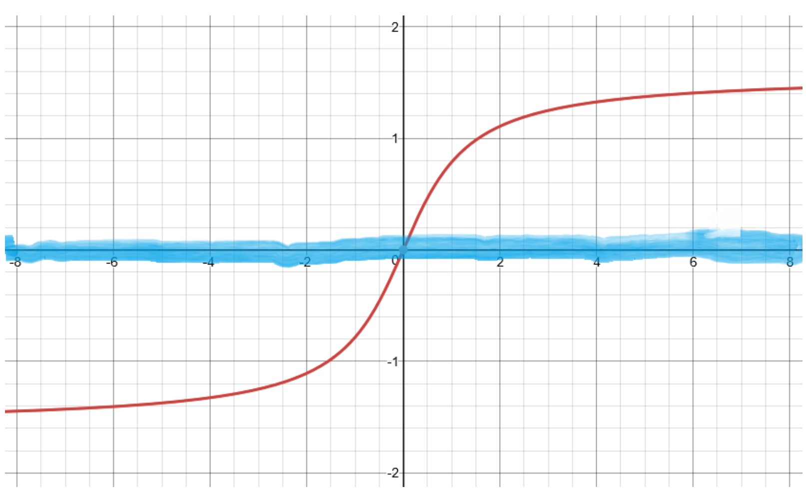
$f(-2)$ and $f(2)$ are both abs maxima and $f(0)$ is an abs min.



$f(x) = x^2$

$I = (-\infty, \infty)$

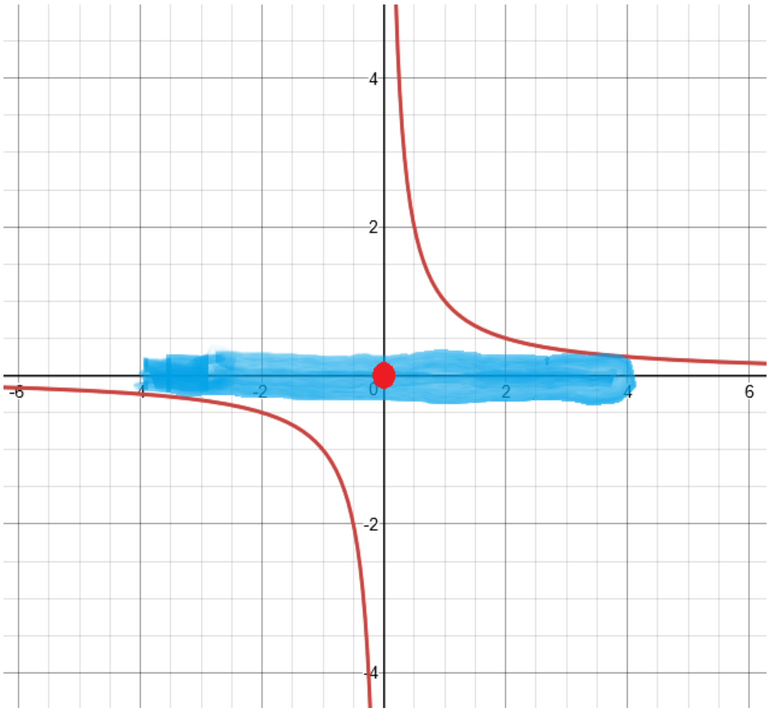
$f(0)$ is still an abs minimum, but there is no abs. maximum on this new I.



$f(x) = \arctan x$

$I = (-\infty, \infty)$

f has neither abs. max nor abs. min on I.



$$f(x)=\begin{cases}1/x & \text{if } x\neq 0\\ 0 & \text{if } x=0\end{cases}$$

$I=[-4,4]$

f has neither abs. max nor abs. min on I.

Since we will be looking for max/min, it will be helpful to know when they exist!

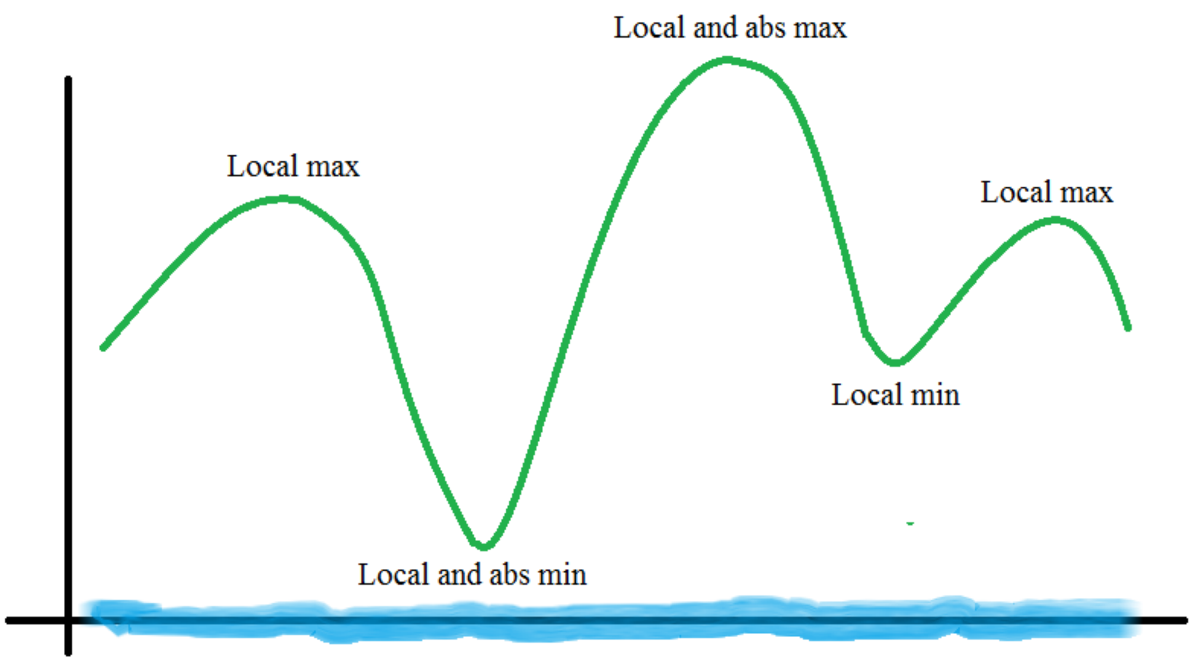
Theorem (Existence of Extrema on a Closed Interval): If f is a continuous function and I is a closed interval, then f has an absolute maximum and absolute minimum on I.

So now we know a situation where max/min exist. But still: how do we find them???

Local Extrema and Critical Points

Definition: We say that f has a local minimum at a if the value of f at a is smaller than the value of f at points nearby a. We say that f has a local maximum at a if the value of f at a is larger than the value of f at points nearby a.

Ex: Imagine the graph of f on I as hilly terrain:

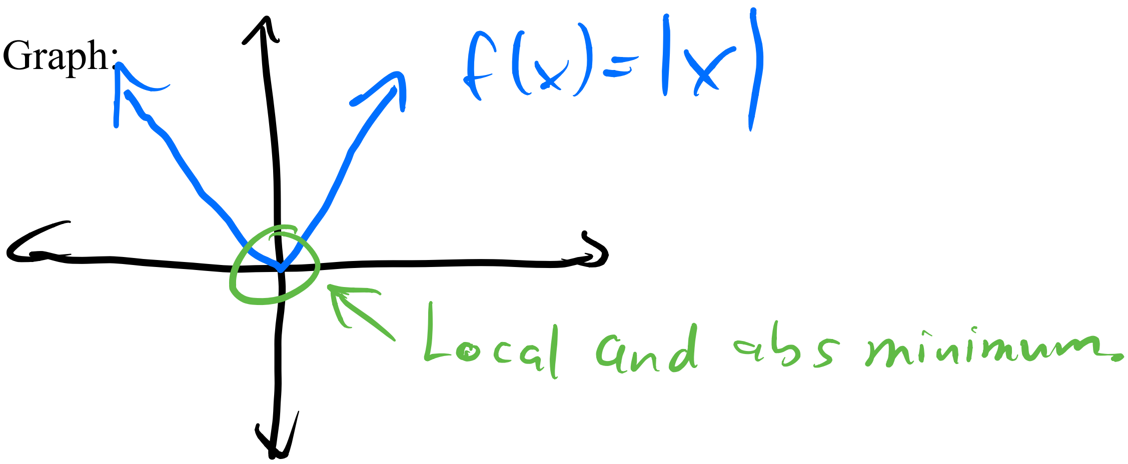


NOTICE: The slopes of the tangent lines at the local extrema are horizontal.

In other words, the derivative evaluated at the local extrema is 0.

It IS possible for a function to have a local extremum at a without having a horizontal tangent line at a. For example, if $f(x) = |x|$, then the graph of f has a local min at $x = 0$.

Graph:



Here the derivative evaluated at the local extremum DNE.

We have seen two possibilities for local extrema:

- Derivative is 0.
- Derivative DNE.

Definition: A number c in the domain of f is called a critical point if either $f'(c) = 0$ or $f'(c)$ DNE.

Ex: Find the critical points of $f(x) = x^3 - 9x^2 + 24x - 10$.

$f'(x) = 3x^2 - 18x + 24$

Solve: $0 = 3x^2 - 18x + 24$

$$x = \frac{18 \pm \sqrt{18^2 - 4 \cdot 3 \cdot 24}}{6} = \frac{18 \pm \sqrt{324 - 288}}{6}$$

$$= \frac{18 \pm 6}{6} = 3 \pm 1$$

$$= \boxed{2, 4}$$

The points $x = 2$ and $x = 4$ are the critical pts.

Ex: Find the critical points of $f(x) = \sin(x^{1/3})$ in the interval $[-1.4, 1.4]$.

$$f'(x) = \cos(x^{1/3})\left(\frac{1}{3}x^{-2/3}\right)$$

$$= \frac{1}{3}\frac{1}{x^{2/3}}\cos(x^{1/3})$$

Set $= 0$ and solve.

$$0 = \frac{1}{3}\frac{1}{x^{2/3}}\cos(x^{1/3}) \Rightarrow 0 = \cos(x^{1/3})$$

$$\Rightarrow x^{1/3} = \ldots, \frac{-3\pi}{2}, \frac{-\pi}{2}, \frac{\pi}{2}, \frac{3\pi}{2}, \ldots$$

$$\Rightarrow x = \ldots \left(\frac{-3\pi}{2}\right)^3, \left(\frac{-\pi}{2}\right)^3, \left(\frac{\pi}{2}\right)^3, \left(\frac{3\pi}{2}\right)^3, \ldots$$

None are in the interval $[-1.4, 1.4]$.

But $f'(x)$ DNE when $x = 0$. So

$x = 0$ is the only critical pt in the interval.

Theorem (How to Find Extreme Values): Let f be a *continuous* function defined on a *closed* interval $I=[a,b]$.

- If f on I has an absolute extremum at the point c, then either c is a critical point, or c is one of the end points, $c=a$ or $c=b$.

Ex: Find the absolute maxima and minima of $f(x)=x^3-9x^2+24x-10$ on the interval $[0,\ 4.5]$.